Pitch Factor Analysis for Symmetric and Asymmetric Tooth Gears

By A.L. Kapelevich

Pitch factor analysis is an analytical tool that can be used for comparison of different gear geometry solutions by exploring the characteristics of involute gear mesh parameters that define gear drive performance.

IN COMPARISON WITH TRADITIONAL GEAR DESIGN based on pre-selected, typically standard generating rack parameters and its addendum modification also known as the X-shift, the alternative Direct Gear Design® method [1, 2] provides certain advantages for custom high-performance gear drives that include increased load capacity, efficiency, and lifetime and reduced size, weight, noise, vibrations, and cost.

Pitch factor analysis is one of Direct Gear Design's methodical approaches to describe the involute gear mesh geometry and explore its characteristics. It divides the operating circular pitch of the involute gear mesh into three segments: the driving or load (or motion) transmitting segment related to the drive tooth flanks, the coast segment that may transmit load (or motion) in reverse related to the coast tooth flanks, and the noncontact segment that is excluded from load (or motion) transmissions related to the tooth tip lands and radii. Ratios of these segments to the operating circular pitch are called the pitch factors. Combinations of these factors greatly affect involute gear mesh parameters that define gear drive performance.

This paper introduces an analytical approach that describes main gear mesh characteristics such as operating pressure angles and contact ratios as functions of the pitch factors. It also considers areas of existence of involute gear pairs with the given constant values of the pitch factors.

DEFINITION OF GEAR TOOTH PROFILES

The Direct Gear Design method does not use pre-selected basic or generating rack to define the gear geometry. Two involute curves unwound from the base circle, the arc distance between them, and tooth tip circle describe a gear tooth profile (see Figure 1). The equally spaced teeth form the gear. The root fillet profile connecting neighboring tooth flanks is not in contact with the mating gear teeth. However, this portion of the tooth profile is critical because this is the area of the bending stress concentration. It is designed to exclude any kind of interference with the mating tooth tip and minimize bending stress.

Similarly, Figure 2 describes an asymmetric gear tooth profile. Asymmetric teeth are beneficial for mostly unidirectional gear transmissions where one (drive) tooth flank carries higher load and a longer period of time than the opposite (coast) tooth [2, 3]. Asymmetric tooth flanks are formed by two involute curves unwound from two different base circles for drive and coast flanks. The design intent of asymmetric tooth gear design is to improve drive tooth flank performance on account of less-loaded coast tooth flank.

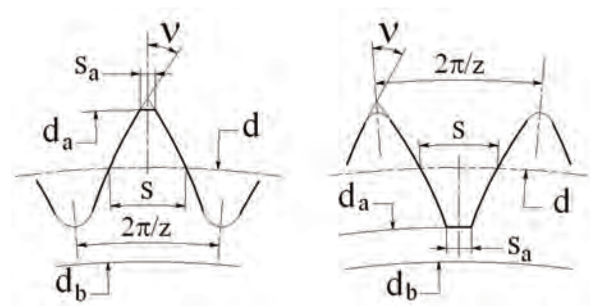

Figure 1: Tooth profile; a – external tooth; b – internal tooth; n – number of teeth; d_a – tooth tip circle diameter; d_b – base circle diameter; d – reference circle diameter; S – circular tooth thickness at the reference diameter; ν – involute intersection profile angle; S_a – circular tooth thickness at the tooth tip diameter

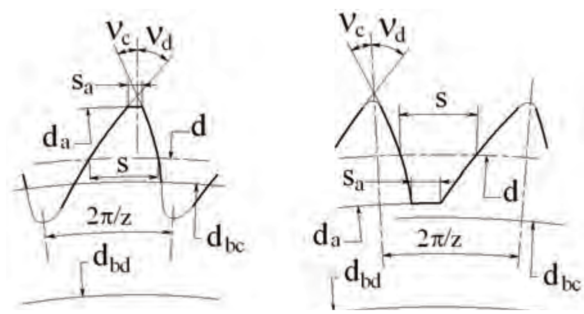

Figure 2: Asymmetric tooth profile; a – external tooth; b – internal tooth; subscripts "d" and "c" are for the drive and coast flanks of the asymmetric tooth

DEFINITION OF PITCH FACTORS

In order to maximize gear drive performance, Direct Gear Design defines the gear tooth geometry elements (flanks, tooth tips, and root fillet) independently. Such approach allows the gear parameters' range to expand beyond the standard gear design limitations. Pitch factor analysis helps to understand relations and limits between such critical gear mesh parameters as operating pressure angle and contact ratio that define the gear tooth flank durability and bending strength.

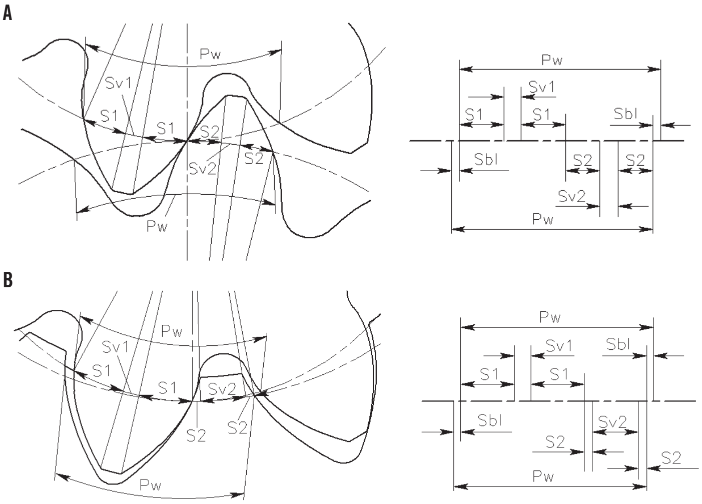

Figure 3: Symmetric tooth gear mesh and operating pitch components; a – external gear mesh; b – internal gear mesh

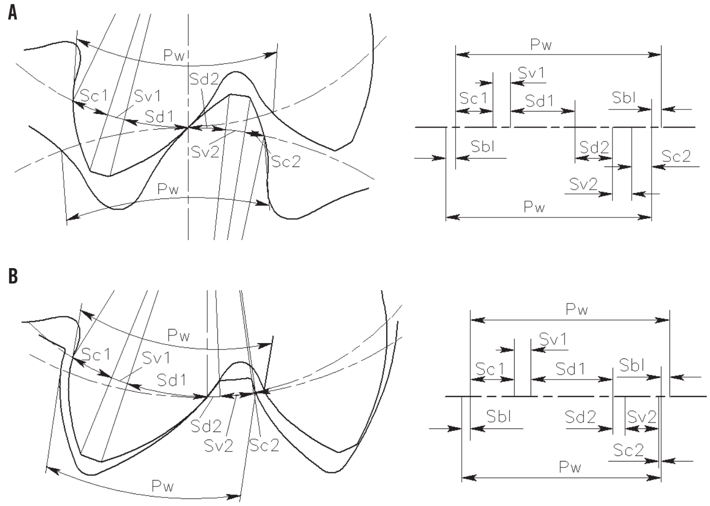

Figure 4: Asymmetric tooth gear mesh and operating pitch components; a – external gear mesh; b – internal gear mesh

The gear mesh operating circular pitch can be presented as:

$$p_w = \frac{\pi \times d_{w1,2}}{z_{1,2}} = S_{w1} + S_{w2} + S_{bl} \qquad \text{Equation 1}$$

where:
indexes 1 and 2 are for mating pinion and gear accordingly;
z_1 and z_2 are numbers of teeth;
d_{w1} and d_{w2} are operating (rolling) pitch diameters equal $d_{w1} = 2a_w/(1+u)$, $d_{w2} = ud_{w1}$;
a_w is a center distance;
$u = z_2/z_1$ is gear ratio;
S_{w1} and S_{w2} are the pinion and gear tooth thicknesses at the operating pitch diameter;
S_{bl} is arc backlash.

Figures 3 and 4 show the symmetric and asymmetric tooth gear mesh and operating pitch components.

The following equations describe the most general case of asymmetric tooth gears. For symmetric tooth gears, these equations are simplified because the drive and coast gear flank parameters are identical.

The tooth thicknesses S_{w1} and S_{w2} from Figure 2 are:

$$S_{w1,2} = S_{d1,2} + S_{c1,2} + S_{v1,2} \quad \textbf{Equation 2}$$

where:
$S_{d1,2}$ are projections of the addendum portion of the drive involute flank on the pitch circle:

$$S_{d1,2} = d_{w1,2}(\pm inv(\alpha_{ad1,2}) \mp inv(\alpha_{wd}))/2 \quad \textbf{Equation 3}$$

α_{wd} – drive flank operating pressure angle equal α_{wd} = arccos $d_{bd1,2}/d_{w1,2}$;
$\alpha_{ad1,2}$ – drive flank involute angles at the tooth tips that are equal $\alpha_{ad1,2}$ = arccos $d_{bd1,2}/d_{a1,2}$.
inv(x)=tan(x)-x – involute function of the x (in radians),
signs ∓ and ± denote – the top sign is for external gear mesh and the bottom sign is for internal gear mesh;
$S_{c1,2}$ are projections of the addendum portion of the coast involute flank on the pitch circle:

$$S_{c1,2} = d_{w1,2}(\pm inv(\alpha_{ac1,2}) \mp inv(\alpha_{wc}))/2 \quad \textbf{Equation 4}$$

α_{wc} – coast flank operating pressure angle equal α_{wc} = arccos $d_{bc1,2}/d_{w1,2}$;
$\alpha_{ac1,2}$ – coast flank involute angles at the tooth tips that are equal $\alpha_{ac1,2}$ = arccos $d_{bc1,2}/d_{a1,2}$;
$S_{v1,2}$ are projections of the tooth tip lands on the pitch circle:

$$S_{v1,2} = S_{a1,2}\frac{d_{w1,2}}{d_{a1,2}} = S_{a1,2}\frac{\cos\alpha_{ad1,2}}{\cos\alpha_{wd}} = S_{a1,2}\frac{\cos\alpha_{ac1,2}}{\cos\alpha_{wc}} \quad \textbf{Equation 5}$$

Then, the gear mesh operating circular pitch for asymmetric tooth gears from Equation 1 is:

$$p_w = S_{d1} + S_{d2} + S_{c1} + S_{c2} + S_{v1} + S_{v2} + S_{bl} \quad \textbf{Equation 6}$$

A pitch factor equation is a result of the division of Equation 6 by operating circular pitch p_w:

$$\theta_d + \theta_c + \theta_v = 1 \quad \textbf{Equation 7}$$

where:
θ_d is the drive pitch factor defined as:

$$\theta_d = (S_{d1} + S_{d2})/p_w \quad \textbf{Equation 8}$$

θ_c is the coast pitch factor defined as:

$$\theta_c = (S_{c1} + S_{c2})/p_w \quad \textbf{Equation 9}$$

and the θ_v is the non-contact pitch factor defined as:

$$\theta_v = (S_{v1} + S_{v2} + S_{bl})/p_w \quad \textbf{Equation 10}$$

The drive pitch factor is:
for the external gear mesh:

$$\theta_d = \frac{z_1}{2\pi}(inv(\alpha_{ad1}) + u\,inv(\alpha_{ad2}) - (1+u)inv(\alpha_{wd})) \quad \textbf{Equation 11}$$

for the internal gear mesh:

$$\theta_d = \frac{z_1}{2\pi}(inv(\alpha_{ad1}) - u\,inv(\alpha_{ad2}) + (u-1)inv(\alpha_{wd}))$$ **Equation 12**

The coast pitch factor is:
for the external gear mesh:

$$\theta_c = \frac{z_1}{2\pi}(inv(\alpha_{ac1}) - u\,inv(\alpha_{ac2}) + (u-1)inv(\alpha_{wc}))$$ **Equation 13**

for the internal gear mesh:

$$\theta_c = \frac{z_1}{2\pi}(inv(\alpha_{ac1}) + u\,inv(\alpha_{ac2}) - (1+u)inv(\alpha_{wc}))$$ **Equation 14**

Assuming the arc backlash $S_{bl} = 0$, the non-contact pitch factor can be presented as:

$$\theta_v = \frac{m_{a1}\cos\alpha_{ad1} + m_{a2}\cos\alpha_{ad2}}{\pi\cos\alpha_{wc}}$$ **Equation 15**

where:
$m_{a1,2} = S_{a1,2}/m_w$ is relative tooth tip thicknesses;
m_w is operating module.

Then, the drive and coast pressure angles can be defined by the following equations:
for the external gear mesh:

$$inv(\alpha_{wd}) = \frac{1}{1+u}(inv(\alpha_{ad1}) + u\,inv(\alpha_{ad2}) - \frac{2\pi\theta_d}{z_1})$$ **Equation 16**

$$inv(\alpha_{wc}) = \frac{1}{1+u}(inv(\alpha_{ac1}) + u\,inv(\alpha_{ac2}) - \frac{2\pi\theta_c}{z_1})$$ **Equation 17**

for internal gear mesh:

$$inv(\alpha_{wd}) = \frac{1}{u-1}(\frac{2\pi\theta_d}{z_1} - inv(\alpha_{ad1}) + u\,inv(\alpha_{ad2}))$$ **Equation 18**

$$inv(\alpha_{wc}) = \frac{1}{u-1}(\frac{2\pi\theta_c}{z_1} - inv(\alpha_{ac1}) + u\,inv(\alpha_{ac2}))$$ **Equation 19**

The drive and coast contact ratios are:
for the external gear mesh:

$$\varepsilon_{\alpha d} = \frac{z_1}{2\pi}(\tan\alpha_{ad1} + u\tan\alpha_{ad2} - (1+u)\tan\alpha_{wd})$$ **Equation 20**

$$\varepsilon_{\alpha c} = \frac{z_1}{2\pi}(\tan\alpha_{ac1} + u\tan\alpha_{ac2} - (1+u)\tan\alpha_{wc})$$ **Equation 21**

for internal gear mesh:

$$\varepsilon_{\alpha d} = \frac{z_1}{2\pi}(\tan\alpha_{ad1} - u\tan\alpha_{ad2} + (u-1)\tan\alpha_{wd})$$ **Equation 22**

$$\varepsilon_{\alpha c} = \frac{z_1}{2\pi}(\tan\alpha_{ac1} - u\tan\alpha_{ac2} + (u-1)\tan\alpha_{wc})$$ **Equation 23**

For symmetric tooth gears, the pitch factor θ from Equation 7 is:

$$\theta = \theta_d = \theta_c = \frac{1}{2}\times(1-\theta_v)$$ **Equation 24**

This equation shows that for symmetric gears, the pitch factor is always ≤ 0.5. For the standard 20° pressure angle gears, θ = 0.25–0.30, and for the 25° pressure angle gears, θ = 0.30–0.35. In custom symmetric gears, the pitch factor θ can reach values of 0.40–0.45. The pitch factor θ = 0.5 is practically not possible if the mating gears do not have the pointed tooth tips.

For gears with asymmetric teeth, the drive pitch factor θ_d from Equation 7 is:

$$\theta_d = 1 - \theta_c - \theta_v$$ **Equation 25**

Reduction of the coast pitch factor θ_c and the non-contact pitch factor θ_v allows a significant increase of the drive pitch factor θ_d. A practical range of the drive pitch factor θ_d varies between 0.40 and 0.6. Although, in theory, it could be close to 1.0 for irreversible asymmetric gears with extremely low numbers of teeth. Examples of such gear profiles and data are shown in Figure 5 and Table 1.

Number of teeth, $z_{1,2}$	1	2
Drive flank pitch factor, θ_d	0.952	0.84
Drive flank pressure angle, α_{wd}°	72.3	57.5
Drive flank contact ratio, $\varepsilon_{\alpha d}$	1.0	1.0
Drive flank tooth tip profile angle, $\alpha_{ad1,2}$°	81.0	72.3

Table 1

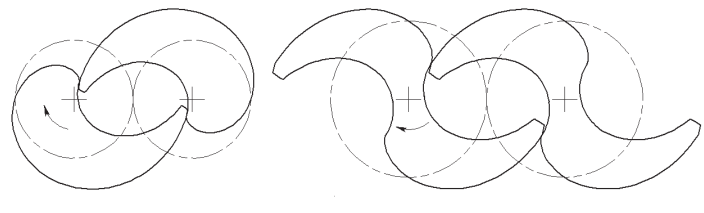

Figure 5: External irreversible asymmetric spur gear profiles with an extremely low number of teeth: a – $z_{1,2}$ = 1; b – $z_{1,2}$ = 2

Figure 6 presents a sample of the drive pressure angle versus the drive contact ratio $\alpha_{wd} - \varepsilon_{\alpha d}$ chart at different values of θ_d and the mesh images for gear pairs with numbers of teeth z_1 = 21 for the pinion and z_2 = 37 for the gear. This chart shows that the symmetric gear solutions lay below the curve θ_d = 0.5, and the asymmetric gear meshes are located below and above this curve.

AREA OF EXISTENCE AND PITCH FACTORS

Direct Gear Design presents the area of existence of asymmetric tooth

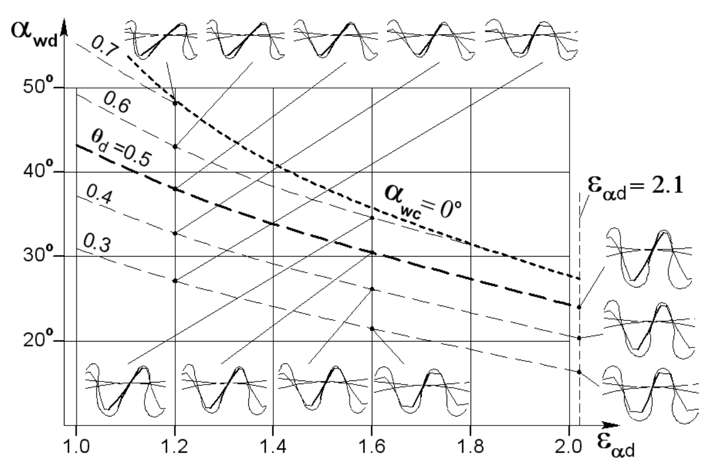

Figure 6: A sample of the $\alpha_{wd} - \varepsilon_{\alpha d}$ chart with different values of the drive pitch factor θ_d for a gear pair with the pinion number of teeth z_1 = 21 and the gear number of teeth z_2 = 37. The non-contact pitch factor θ_v ≥ 0.15. The tooth tip thicknesses of mating gears are assumed equal, $S_{a1} = S_{a2}$

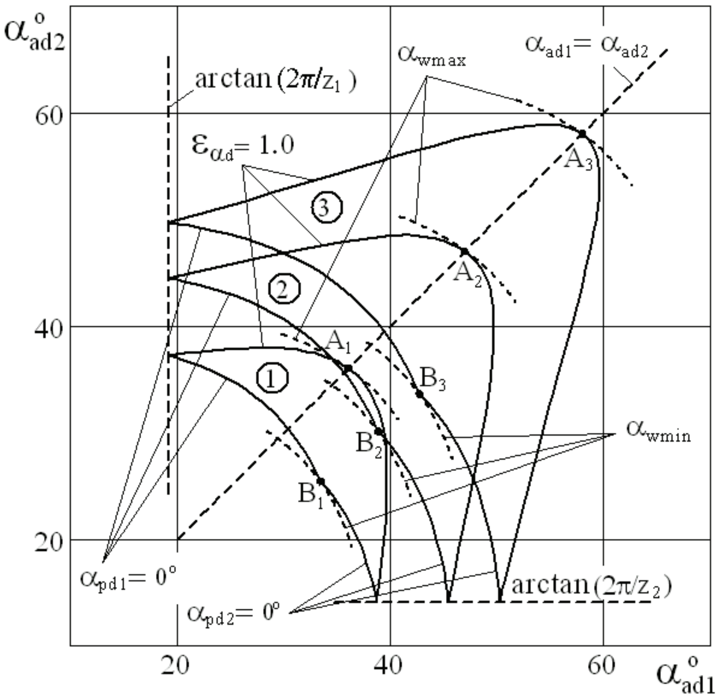

Figure 7: Areas of existence of external spur gears with z_1 = 18, z_2 = 25, and different values of the pitch factor θ_d; 1 – θ_d = 0.3; 2 – θ_d = 0.5; 3 – θ_d = 0.7

gears with the given numbers of teeth z_1 and z_2, constant asymmetry factor K = cos α_{wc}/cos α_{wd}, and relative tooth tip thicknesses m_{a1} and m_{a2}. [2, 4]. The pitch factors θ_d, θ_c, and θ_v in such areas of existence are varying. Figure 7 presents the overlaid areas of existence of spur external gears with the different constant drive flank pitch factors θ_d. This type of area of existence of involute gears defines only the drive flank gear meshes. If $\theta_d \leq 0.5$, the gears can have symmetric or asymmetric teeth. If $\theta_d > 0.5$, the gears can have only asymmetric teeth. The gears with symmetric teeth are always reversible. The gears with asymmetric teeth can be reversible or irreversible depending on the coast flank pitch factor θ_c selection.

The areas of existence in Figure 7 is limited by the interference isograms $\alpha_{pd1} = 0°$ and $\alpha_{pd2} = 0°$ that describe a beginning of undercut of the involute drive flank near the root fillet by the mating tooth tip and the isogram $\varepsilon_{\alpha d} = 1.0$, the minimal value of the drive flank transverse contact ratio for spur gears.

The interference isograms $\alpha_{pd1} = 0°$ and $\alpha_{pd2} = 0°$ are defined by Equation 16 and the following equations [1]:

$$u\tan\alpha_{ad2} - (1+u)\tan\alpha_{wd} = 0 \qquad \text{Equation 26}$$

and

$$\tan\alpha_{ad1} - (1+u)\tan\alpha_{wd} = 0 \qquad \text{Equation 27}$$

respectively.

The minimal for spur gears' contact ratio isogram $\varepsilon_{\alpha d} = 1.0$ is defined by Equations 16 and 20.

In point A of the area of existence where the drive flank pressure angle α_{wd} is maximum and the contact ratio $\varepsilon_{\alpha d} = 1.0$, the pressure angle and contact ratio isograms have a common tangent point, and the first derivatives of these isogram functions should be equal:

$$\frac{d(\alpha_{ad2})}{d(\alpha_{ad1})}\bigg|_{\alpha_{wd}=const} = \frac{d(\alpha_{ad2})}{d(\alpha_{ad1})}\bigg|_{\varepsilon_{\alpha d}=1.0} \qquad \text{Equation 28}$$

or $\alpha_{ad1} = \alpha_{ad2}$.

This means that the points A of the areas of existence lay on the straight line $\alpha_{ad1} = \alpha_{ad2}$. The pressure angle equation at point A is defined as a solution of Equations 16 and 20.

$$\tan\alpha_{wd} + \frac{2\pi}{(1+u)z_1} - \tan\left(\alpha_{wd} + \frac{2\pi(1-\theta_d)}{(1+u)z_1}\right) = 0 \qquad \text{Equation 29}$$

Its solution is [2]:

$$\alpha_{wd}^{A} = \arctan\left(\sqrt{\frac{\pi^2}{(1+u)^2 z_1^2} + \frac{2\pi}{z_t \tan\frac{2\pi(1-\theta_d)}{(1+u)z_1}} - 1} - \frac{\pi}{(1+u)z_1}\right) \qquad \text{Equation 30}$$

In point B at the intersection of the interference isograms $\alpha_{pd1} = 0°$ and $\alpha_{pd2} = 0°$, the pressure angle is minimum and the contact ratio $\varepsilon_{\alpha d}$ is maximum. This maximum contact ratio is defined as a solution of Equations 16, 20, 26, and 27:

$$\arctan\left(\frac{2\pi\varepsilon_{\alpha d}}{z_1}\right) + u\arctan\left(\frac{2\pi\varepsilon_{\alpha d}}{z_2}\right) - (1+u)\arctan\left(\frac{2\pi\varepsilon_{\alpha d}}{z_t}\right) - \frac{2\pi(\varepsilon_{\alpha d}-\theta_d)}{z_1} = 0 \qquad \text{Equation 31}$$

The drive flank pressure angle α_{wd} at point B is [1]:

$$\alpha_{wd}^{B} = \arctan\left(\frac{2\pi\varepsilon_{\alpha d}}{(1+u)z_1}\right) \qquad \text{Equation 32}$$

The pressure angles α_{wd} and contact ratios $\varepsilon_{\alpha d}$ at points A and B of the areas of existence from Figure 7 are presented in Table 2.

Points of the area of existence with the constant drive flank pitch factor do not define complete mating gear teeth, but just their drive flanks. This allows independently selecting the tooth tip thicknesses

Drive Flank Pitch Factor θ_d		0.3	0.5	0.7
Point A	α_{wd},°	30.19	42.86	55.51
	$\varepsilon_{\alpha d}$	1.0	1.0	1.0
Point B	α_{wd},°	15.46	18.64	21.14
	$\varepsilon_{\alpha d}$	1.89	2.31	2.65

Table 2

and the coast tooth flank parameters of asymmetric gears.

When some point of the area of existence with the coordinates α_{ad1} and α_{ad2} is chosen, the pressure angle α_{wd} is calculated by Equation 16. Then, after selection of the relative tooth tip thicknesses m_{a1} and m_{a2}, the non-contact pitch factor θ_v is calculated by Equation 15. This allows the coast flank pitch factor to be defined from Equation 7:

$$\theta_c = 1 - \theta_d - \theta_v$$

Equation 33

If the tooth tip radii are equal to zero, the asymmetry factor K can be defined as a solution of equations $K = \cos\alpha_{wc}/\cos\alpha_{wd} = \cos\alpha_{ac1,2}/\cos\alpha_{ad1,2}$ and Equation 16:

$$(1+u)inv(arc\cos(K\cos\alpha_{wd})) = inv(arc\cos(K\cos\alpha_{ad1})) + u\,inv(arc\cos(K\cos\alpha_{ad2})) - \frac{2\pi\theta_c}{z_1}$$

Equation 34

Then, the coast flank pressure angle can be defined:

$$\alpha_{wc} = arc\cos(K\cos\alpha_{wd})$$

Equation 35

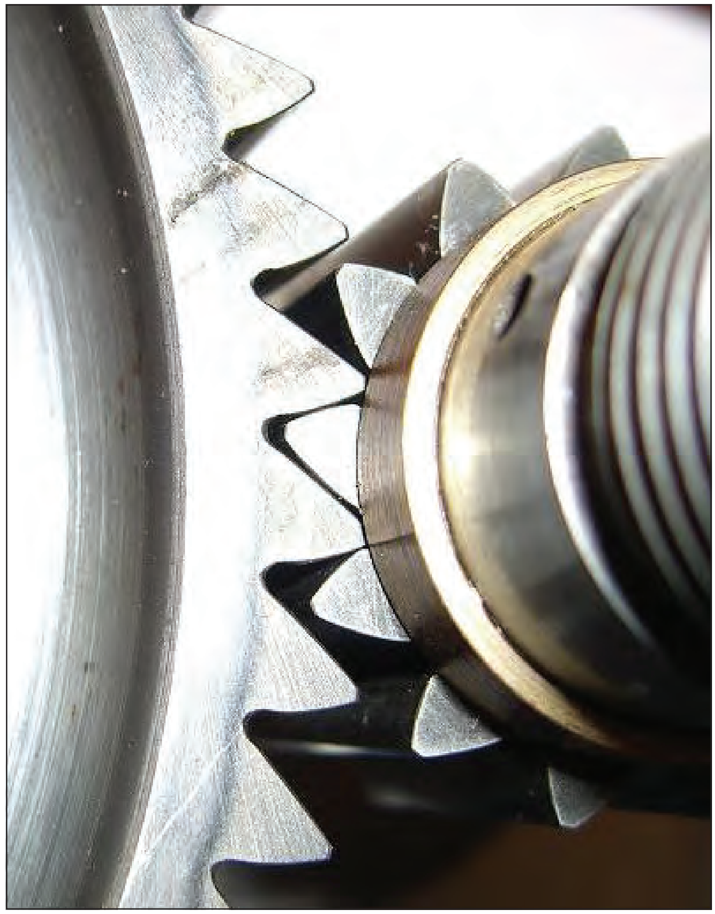

Figure 8: Experimental asymmetric tooth gear set

The coast flank contact ratio is defined by Equation 21.

EXAMPLE OF APPLICATION

Figure 8 presents an experimental asymmetric tooth spur gear set of an electric generator driven by the 9I56 gas turbine engine. Table 3 show the main tooth geometry data of this gear set.

This gear set has a high drive pitch factor θ_d = 0.58 that is not achievable for gears with symmetric teeth. In comparison testing with the baseline helical gear set designed by standards, the experimental asymmetric tooth spur gear set demonstrated significant stress and a vibration level reduction [3].

SUMMARY

A simultaneous increase of the drive pressure angle and the drive contact ratio maximizes gear drive performance. It allows reducing the contact and bending stress and increasing load capacity and power transmission density. This indicates potential advantages of the directly designed asymmetric tooth gears over the symmetric ones for gear drives that transmit load mostly in one direction.

The pitch factor analysis is the additional Direct Gear Design analytical tool that can be used for comparison of different gear geometry solutions, helping the designer better understand the available options and choose the optimal one.

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	Pinion	Gear
Number of teeth z_1 and z_2	13	60
Operating Module m, mm	2.034	
Center Distance a_w, mm	74.25	
Drive Flank Operating Pressure Angle α_d,°	41.00	
Coast Flank Operating Pressure Angle α_c,°	18.00	
Drive Flank Contact Ratio $\varepsilon_{\alpha d}$	1.20	
Coast Flank Contact Ratio $\varepsilon_{\alpha c}$	1.64	
Asymmetry Factor K	1.26	
Drive Pitch Factor θ_d	0.58	
Coast Pitch Factor θ_c	0.277	
Non-Contact Pitch Factor θ_v	0.143	

Table 3